Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

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Mathematics: applications and interpretation

Higher level

Paper 3

16 pages

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Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.
A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.
R Marks awarded for clear **Reasoning**.
AG Answer given in the question and so no marks are awarded.
FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1***, ***A2***.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part. Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g.** ***(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.
- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.
- If the error leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.
- If the candidate’s answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a “show that” question correctly, and then in subsequent parts use their incorrect answer rather than the given value.
- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.
- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was “Hence”.

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).
- Miscopying of candidates' own work does **not** constitute a misread, it is an error.
- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.
- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1{,}9$ or 1000 and $1{,}000$ and 1.000.
- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but If you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is "first".

1. (a) (i) Any one from: ***R1***
increase sample size / increase response rate / repeat process
check whether sample is representative
test-retest participants or do a parallel test
use a stratified sample
use a random sample

Note: Do not condone:
Ask different types of doctor
Ask for proof of income
Ask for proof of being a doctor
Remove anonymity
Remove response K.

[1 mark]

(ii) Any one from: ***R1***
non-random sampling means a subset of population might be responding
self-reported happiness is not the same as happiness
happiness is not a constant / cannot be quantified / is difficult to measure
income might include external sources
Juliet is only sampling doctors in her city
correlation does not imply causation
sample might be biased

Note: Do not condone the following common but vague responses unless they make a clear link to validity:
Sample size is too small
Result is not generalizable
There may be other variables Juliet is ignoring
Sample might not be representative

[1 mark]

(b) because the income is very different / implausible / clearly contrived ***R1***

Note: Answers must explicitly reference "income" to get credit.

[1 mark]

continued…

Question 1 continued

(c) (i) $(\$)\ 90\ 200$ ***(M1)A1***

[2 marks]

(ii) $r = 0.558\ \ (0.557723...)$ ***A2***

[2 marks]

(d) (i) **EITHER**

only looking for change in one direction ***R1***

OR

only looking for greater happiness with greater income ***R1***

OR

only looking for evidence of positive correlation ***R1***

[1 mark]

(ii) $H_0 : \rho = 0;\ H_1 : \rho > 0$ ***A1A1***

Note: Award ***A1*** for ρ seen (do not accept r), ***A1*** for both correct hypotheses, using **their** ρ or r. Accept an equivalent statement in words, however reference to "correlation for the population" or "association for the population" must be explicit for the first ***A1*** to be awarded.

Watch out for a null hypothesis in words similar to "Annual income is not associated with greater happiness". This is effectively saying $\rho \leq 0$ and should not be condoned.

[2 marks]

continued…

Question 1 continued

(iii) **METHOD 1 – using critical value of *r***

$0.558 > 0.549$ $(0.557723... > 0.549)$ ***R1***

(therefore significant evidence of) a positive correlation ***A1***

Note: Do not award ***R0A1***.

METHOD 2 – using *p*-value

$0.0469 < 0.05$ $(0.0469463... < 0.05)$ ***A1***

Note: Follow through from their *r*-value from part (c)(ii).

(therefore significant evidence of) a positive correlation ***A1***

Note: Do not award ***A0A1***.

[2 marks]

(e) (i) $a = 0.000126$ $(0.000125842...)$, $b = 41.1$ $(41.1490...)$ ***A1***

[1 mark]

(ii) **EITHER**

the amount the happiness score increases for every \$1 increase in (annual) income ***A1***

OR

rate of change of happiness with respect to (annual) income ***A1***

Note: Accept equivalent responses e.g. an increase of 1.26 in happiness for every $\$10000$ increase in salary.

[1 mark]

(iii) $c = -2.06 \times 10^{-9}$ $(-2.06191... \times 10^{-9})$,

$d = 7.05 \times 10^{-4}$ $(7.05272... \times 10^{-4})$,

$e = 12.6$ $(12.5878...)$ ***A1***

[1 mark]

continued…

Question 1 continued

(iv) for quadratic model: $R^2 = 0.659 \ (0.659145...)$ ***A1***

for linear model: $R^2 = 0.311 \ (0.311056...)$ ***A1***

Note: Follow through from their r value from part (c)(ii).

[2 marks]

(v) **EITHER**

quadratic model is a better fit to the data / more accurate ***A1***

OR

quadratic model explains a higher proportion of the variance ***A1***

[1 mark]

(vi) **EITHER**

not valid, R^2 not a useful measure to compare models with different numbers of parameters ***A1***

OR

not valid, quadratic model will always have a better fit than a linear model ***A1***

Note: Accept any other sensible critique of the validity of the method. Do not accept any answers which focus on the conclusion rather than the method of model selection.

[1 mark]

continued…

Question 1 continued

(f) (i) (single sample) t-test ***A1***

[1 mark]

(ii) **EITHER**

$H_0 : \mu = 80\,000;\ H_1 : \mu \neq 80\,000$ ***A1***

OR

H_0 : (sample is drawn from a population where) the population mean is $\$80\,000$

H_1 : the population mean is not $\$80\,000$ ***A1***

Note: Do not allow ***FT*** from an incorrect test in part (f)(i) other than a z-test.

[1 mark]

(iii) $p = 0.610\ \ (0.610322...)$ ***A1***

Note: For a z-test follow through from part (f)(i), either 0.578 (from biased estimate of variance) or 0.598 (from unbiased estimate of variance).

$0.610 > 0.05$ ***R1***

EITHER

no (significant) evidence that mean differs from $\$80\,000$ ***A1***

OR

the sample could plausibly have been drawn from the quoted population ***A1***

Note: Allow ***R1FTA1FT*** from an incorrect p-value, but the final ***A1*** must still be in the context of the original research question.

[3 marks]

[Total 24 marks]

2. (a) (i) population growth rate / birth rate of sharks (due to eating mackerel) ***A1***

[1 mark]

(ii) (net) death rate of sharks ***A1***

[1 mark]

(b) (i) $\gamma MS - \delta S = 0$ ***A1***

since $S \neq 0$ ***R1***

Note: Accept $S > 0$.

getting to given answer without further error by either cancelling or factorizing ***A1***

$M = \frac{\delta}{\gamma}$ ***AG***

[3 marks]

(ii) $\frac{\mathrm{d}M}{\mathrm{d}t} = 0$

$\alpha M - \beta MS = 0$ ***(M1)***

(since $M \neq 0$) $S = \frac{\alpha}{\beta}$ ***A1***

[2 marks]

(c) (i) $M_{eq} = \frac{\delta}{\gamma} \Rightarrow \frac{\delta}{\frac{1}{2}\gamma} = 2M_{eq}$ ***M1***

Note: Accept equivalent in words.

Doubles ***A1***

Note: Do not accept "increases".

[2 marks]

(ii) $M_{eq} = \frac{\delta}{\gamma}$ is not dependent on α ***R1***

Note: Award ***R0*** for any contextual argument.

no change ***A1***

Note: Do not award ***R0A1***.

[2 marks]

continued…

Question 2 continued

(d) (i) $\frac{\mathrm{d}M}{\mathrm{d}t} = \alpha M$ ***A1***

[1 mark]

(ii) $\int \frac{1}{M}\,\mathrm{d}M = \int \alpha\,\mathrm{d}t$ ***M1***

Note: Award ***M1*** is for an attempt to separate variables. This means getting to the point $\int f(M)\,\mathrm{d}M = \int g(t)\,\mathrm{d}t$ where the integral can be seen or implied by further work.

$\ln|M| = \alpha t + c$ ***A1***

Note: Accept $\ln M$. Condone missing constant of integration for this mark.

$M = k\mathrm{e}^{\alpha t}$
when $t = 0$, $M_0 = k$ ***M1***

Note: Award ***M1*** for a clear attempt at using initial conditions to find a constant of integration. Only possible if the constant of integration exists. $t = 0$ or "initially" or similar must be seen. Substitution may appear earlier, following the integration.

initial conditions and all other manipulations correct and clearly communicated to get to the final answer ***A1***
$M = M_0\mathrm{e}^{\alpha t}$ ***AG***

[4 marks]

(iii) $M = 3M_0$ seen anywhere ***(A1)***
substituting $t = 2$, $M = 3M_0$ into equation $M = M_0\mathrm{e}^{\alpha t}$ ***(M1)***
$3M_0 = M_0\mathrm{e}^{2\alpha}$
$\alpha = \frac{1}{2}\ln 3$ **OR** $0.549306...$ ***A1***

Note: The ***A1*** requires either the exact answer or an answer to at least 4 sf.

≈ 0.549 ***AG***

[3 marks]

continued…

Question 2 continued

(e) (i) an attempt to set up one recursive equation ***(M1)***

Note: Must include **two** given parameters **and** M_n and S_n **and** M_{n+1} or S_{n+1} for the ***(M1)*** to be awarded.

$M_{n+1} = M_n + 0.1(0.549M_n - 0.236M_nS_n)$ ***A1***

$S_{n+1} = S_n + 0.1(0.244M_nS_n - 1.39S_n)$ ***A1***

[3 marks]

(ii) **EITHER**

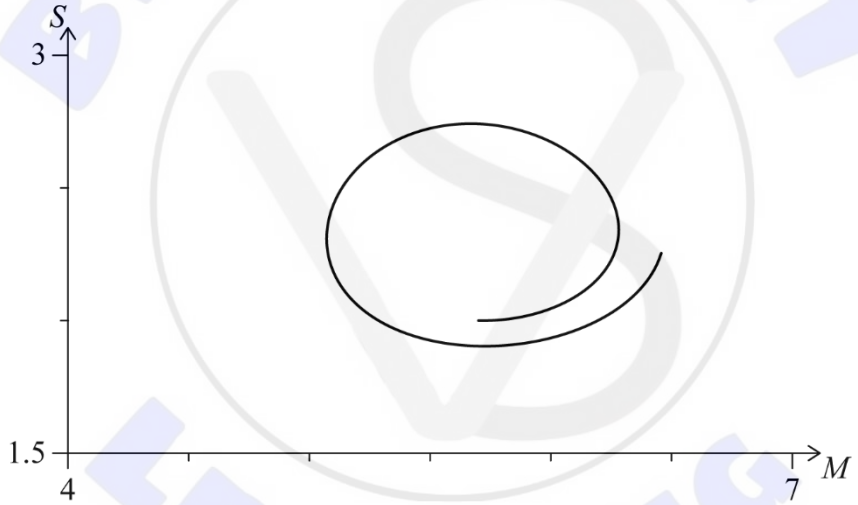
$6.12 \ (6.11609...)$ ***A2***

OR

$6120 \ (6116.09...)$ (mackerel per km^3) ***A2***

[2 marks]

(f) (i)



spiral or closed loop shape ***A1***

approximately 1.25 rotations (can only be awarded if a spiral) ***A1***

correct shape, in approximately correct position (centred at approx. $(5.5, 2.5)$) ***A1***

Note: Award ***A0A0A0*** for any plot of S or M against t.

[3 marks]

continued…

Question 2 continued

(ii) **EITHER**

approximate minimum is $(5.07223\ldots)\ 5.07$ (which is greater than 5) ***A1***

OR

the line $M = 5$ clearly labelled on their phase portrait ***A1***

THEN

(the density will not fall below 5000) hence sufficient for sustainable fishing

A1

Note: Do not award ***A0A1***. Only if the minimum point is labelled on the sketch then a statement here that *"the mackerel population is always above 5000*" would be sufficient. Accept the value 5.07 seen within a table of values.

[2 marks]

(iii) Any two from: ***A1A1***

- Current values / parameters are only an estimate,
- The Euler method is only an approximate method / choosing $h = 0.1$ might be too large.
- There might be random variation / the model has no stochastic component
- Conditions / parameters might change over the nine years,
- A discrete system is being approximated by a continuous system,

Allow any other sensible critique.

If a candidate identifies factors which the model ignores, award ***A1*** per factor identified. These factors could include:

- Other predators
- Seasonality
- Temperature
- The effect of fishing
- Environmental catastrophe
- Migration

Note: Do not allow:
"You cannot have 5.07 mackerel".
It is only a model (as this is too vague).
Some factors have been ignored (without specifically identifying the factors).
Values do not always follow the equation / model. (as this is too vague)

[2 marks]

[Total 31 marks]