2224 – 7102M

Diploma Programme
Programme du diplôme
Programa del Diploma

Markscheme

May 2024

Mathematics: analysis and approaches

Higher level

Paper 2

29 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

R Marks awarded for clear **Reasoning**.

AG Answer given in the question and so no marks are awarded.

FT Follow through. The practice of awarding marks, despite candidate errors in previous parts, for their correct methods/answers using incorrect results.

Using the markscheme

1 General

Award marks using the annotations as noted in the markscheme *eg* ***M1, A2****.*

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is generally not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where there are two or more ***A*** marks on the same line, they may be awarded independently; so if the first value is incorrect, but the next two are correct, award ***A0A1A1***.
- Where the markscheme specifies ***A3***, ***M2*** *etc*., do **not** split the marks, unless there is a note.
- The response to a "show that" question does not need to restate the ***AG*** line, unless a **Note** makes this explicit in the markscheme.
- Once a correct answer to a question or part question is seen, ignore further working even if this working is incorrect and/or suggests a misunderstanding of the question. This will encourage a uniform approach to marking, with less examiner discretion. Although some candidates may be advantaged for that specific question item, it is likely that these candidates will lose marks elsewhere too.
- An exception to the previous rule is when an incorrect answer from further working is used **in a subsequent part**. For example, when a correct exact value is followed by an incorrect decimal approximation in the first part and this approximation is then used in the second part. In this situation, award ***FT*** marks as appropriate but do not award the final ***A1*** in the first part.

Examples:

	Correct answer seen	**Further working seen**	**Any FT issues?**	**Action**
1.	$8\sqrt{2}$	5.65685... *(incorrect decimal value)*	No. Last part in question.	Award ***A1*** for the final mark *(condone the incorrect further working)*
2.	$\frac{35}{72}$	0.468111… (incorrect decimal value)	Yes. Value is used in subsequent parts.	Award ***A0*** for the final mark *(and full **FT** is available in subsequent parts)*

3 Implied marks

Implied marks appear in **brackets e.g. *(M1)***,and can only be awarded if **correct** work is seen or implied by subsequent working/answer.

4 Follow through marks (only applied after an error is made)

Follow through (***FT***) marks are awarded where an incorrect answer from one **part** of a question is used correctly in **subsequent** part(s) (e.g. incorrect value from part (a) used in part (d) or incorrect value from part (c)(i) used in part (c)(ii)). Usually, to award ***FT*** marks, **there must be working present** and not just a final answer based on an incorrect answer to a previous part. However, if all the marks awarded in a subsequent part are for the answer or are implied, then ***FT*** marks should be awarded for *their* correct answer, even when working is not present.

For example: following an incorrect answer to part (a) that is used in subsequent parts, where the markscheme for the subsequent part is ***(M1)A1***, it is possible to award full marks for *their* correct answer, **without working being seen.** For longer questions where all but the answer marks are implied this rule applies but may be overwritten by a **Note** in the Markscheme.

- Within a question part, once an **error** is made, no further ***A*** marks can be awarded for work which uses the error, but ***M*** marks may be awarded if appropriate.

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks, by reflecting on what each mark is for and how that maps to the simplified version.

- If the error leads to an inappropriate value (*e.g*. probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- The markscheme may use the word “their” in a description, to indicate that candidates may be using an incorrect value.

- If the candidate's answer to the initial question clearly contradicts information given in the question, it is not appropriate to award any ***FT*** marks in the subsequent parts. This includes when candidates fail to complete a "show that" question correctly, and then in subsequent parts use their incorrect answer rather than the given value.

- Exceptions to these ***FT*** rules will be explicitly noted on the markscheme.

- If a candidate makes an error in one part but gets the correct answer(s) to subsequent part(s), award marks as appropriate, unless the command term was "Hence".

5 Mis-read

If a candidate incorrectly copies values or information from the question, this is a mis-read (***MR***). A candidate should be penalized only once for a particular misread. Use the ***MR*** stamp to indicate that this has been a misread and do not award the first mark, even if this is an ***M*** mark, but award all others as appropriate.

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.

- If the ***MR*** leads to an inappropriate value (*e.g.* probability greater than 1, $\sin\theta = 1.5$, non-integer value where integer required), do not award the mark(s) for the final answer(s).

- Miscopying of candidates' own work does **not** constitute a misread, it is an error.

- If a candidate uses a correct answer, to a "show that" question, to a higher degree of accuracy than given in the question, this is NOT a misread and full marks may be scored in the subsequent part.

- ***MR*** can only be applied when work is seen. For calculator questions with no working and incorrect answers, examiners should **not** infer that values were read incorrectly.

6 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If the command term is 'Hence' and not 'Hence or otherwise' then alternative methods are not permitted unless covered by a note in the mark scheme.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for parts of questions are indicated by **EITHER** . . . **OR**.

7 Alternative forms

Unless the question specifies otherwise, **accept** equivalent forms.

- As this is an international examination, accept all alternative forms of **notation** for example 1.9 and $1,9$ or 1000 and $1,000$ and 1.000.

- Do not accept final answers written using calculator notation. However, ***M*** marks and intermediate ***A*** marks can be scored, when presented using calculator notation, provided the evidence clearly reflects the demand of the mark.

- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.

- In the markscheme, some **equivalent** answers will generally appear in brackets. Not all equivalent notations/answers/methods will be presented in the markscheme and examiners are asked to apply appropriate discretion to judge if the candidate work is equivalent.

8 Format and accuracy of answers

If the level of accuracy is specified in the question, a mark will be linked to giving the answer to the required accuracy. If the level of accuracy is not stated in the question, the general rule applies to final answers: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures.*

Where values are used in subsequent parts, the markscheme will generally use the exact value, however candidates may also use the correct answer to 3 sf in subsequent parts. The markscheme will often explicitly include the subsequent values that come "*from the use of 3 sf values*".

Simplification of final answers: Candidates are advised to give final answers using good mathematical form. In general, for an ***A*** mark to be awarded, arithmetic should be completed, and any values that lead to integers should be simplified; for example, $\sqrt{\frac{25}{4}}$ should be written as $\frac{5}{2}$. An exception to this is simplifying fractions, where lowest form is not required (although the numerator and the denominator must be integers); for example, $\frac{10}{4}$ may be left in this form or written as $\frac{5}{2}$. However, $\frac{10}{5}$ should be written as 2, as it simplifies to an integer.

Algebraic expressions should be simplified by completing any operations such as addition and multiplication, e.g. $4e^{2x} \times e^{3x}$ should be simplified to $4e^{5x}$, and $4e^{2x} \times e^{3x} - e^{4x} \times e^{x}$ should be simplified to $3e^{5x}$. Unless specified in the question, expressions do not need to be factorized, nor do factorized expressions need to be expanded, so $x(x+1)$ and $x^2 + x$ are both acceptable.

Please note: intermediate ***A*** marks do NOT need to be simplified.

9 Calculators

A GDC is required for this paper, but if you see work that suggests a candidate has used any calculator not approved for IB DP examinations (eg CAS enabled devices), please follow the procedures for malpractice.

10. Presentation of candidate work

Crossed out work: If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work unless an explicit note from the candidate indicates that they would like the work to be marked.

More than one solution: Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise. If the layout of the responses makes it difficult to judge, examiners should apply appropriate discretion to judge which is “first”.

Section A

1. (a) recognition that a 15% loss leaves 85% OR finding 15% and subtracting from original ***(M1)***

0.85×35000 OR $35000 - 0.15 \times 35000$

$= (\$) 29750$ ***A1***

Note: Accept $(\$) 29800$.

[2 marks]

(b) **EITHER**

29750×0.89^{9} ***(A1)***

OR

$\mathrm{N} = 9$
$\mathrm{I\%} = -11$
$\mathrm{PV} = \mp 29750$

(A1)

THEN

value $(FV) = (\$) 10423$ ***A1***

Note: For this ***A1*** the answer must be rounded to the nearest dollar. Accept $(\$) 10441$ from using 3 sf answer from part (a).

[2 marks]

continued…

Question 1 continued

(c) **METHOD 1**

attempt to solve the inequality (or equation) $29750 \times 0.89^{n-1} < 3500$ OR table of values ***(M1)***

$19.3643...$ OR $(n = 19 \Rightarrow)\ 3651.80...$ OR $(n = 20 \Rightarrow)\ 3250.10...$ ***(A1)***

Note: For candidates using $(\$)29800$, $n > 19.3787...$, $(n = 19 \Rightarrow)\ 3657.93...$, $(n = 20 \Rightarrow)\ 3255.56...$.

$n = 20$ ***A1***

[3 marks]

METHOD 2

use of the finance app with $\mathrm{I\%} = -11,\ \mathrm{PV} = \mp 29750,\ \mathrm{FV} = \pm 3500$

OR $29750 \times 0.89^{N} < 3500$ (condone the use of n or x) ***(M1)***

$(N =)\ 18.3643...$ ***(A1)***

Note: For candidates using $(\$)29800$, $N = 18.3787...$

$n = 20$ ***A1***

[3 marks]

Total [7 marks]

2. (a) attempt to use trigonometry to find the radius of the cone OR Oliver's distance from centre $(r+5)$ ***(M1)***

$\tan 58^\circ = \dfrac{18.2}{r+5}$ OR $\dfrac{r+5}{\sin 32^\circ} = \dfrac{18.2}{\sin 58^\circ}$ OR $(r+5=)\,11.3726...$ ***(A1)***

$r = 6.37262...$ (m)

$(r=)\ 6.37$ (m) ***A1***

[3 marks]

(b) attempt to substitute $h = 20$ and their radius into the correct volume of cone formula ***(M1)***

$$V = \frac{\pi(6.37262...)^2(20)}{3}$$

$= 850.540\ldots$

$= 851\ (\text{m}^3)$ ***A1***

Note: Accept $849.840\ldots$ (850) obtained from using $r = 6.37$.

[2 marks]

Total [5 marks]

3. (a) recognition of $X > 13$ OR $Z > 1.5$ (could be seen in a diagram) ***(M1)***

$\left(\mathrm{P}(X > 13) =\right)$ 0.0668072...

$= 0.0668$ ***A1***

[2 marks]

(b) **EITHER**

equating an appropriate correct normal CDF function to 0.1 or 0.9 ***(M1)***

$\mathrm{P}(X > 10 + 2k) = 0.1$ OR $\mathrm{P}(Z < k) = 0.9$ OR $\mathrm{P}(X < 10 - 2k) = 0.1$ OR $\mathrm{P}(Z < -k) = 0.1$

OR

recognising need to use inverse normal with 0.1 or 0.9 ***(M1)***

THEN

1.28155...

$k = 1.28$ ***A1***

[2 marks]

Total [4 marks]

4. (a) recognition that velocity is zero ***(M1)***

$v = 2\sin(0.5t) + 0.3t - 2 = 0$

$t = 1.68694\ldots$

$t = 1.69$ ***A1***

[2 marks]

(b) recognition that $v > 0$ ***(M1)***

$1.68694... < t < 6.11857...$

$1.69 < t < 6.12$ ***A1***

[2 marks]

(c) attempt to substitute into the total displacement formula (condone missing or incorrect limits, and absence of $\mathrm{d}t$) ***(M1)***

$\int_0^{10} \left(2\sin(0.5t) + 0.3t - 2\right)\mathrm{d}t$ OR $\int_0^{10} v(t)\,\mathrm{d}t$

$= -2.13464...$

$= -2.13\,(\mathrm{m})$ ***A1***

Note: Award ***(M1)A0*** if -2.13 is followed by 2.13.

[2 marks]

Total [6 marks]

5. (a) $r = 0.901017\ldots$

$r = 0.901$ ***A2***

[2 marks]

(b) Student 11 Test B: should not extrapolate ***R1***

[1 mark]

(c) (i) Student 12 Test A: should not use line of y on x to predict x from y (or equivalent) ***R1***

(ii) attempt to find the equation of the regression line of x on y ***(M1)***

$(x =)\ 0.987124\ldots y - 3.21970\ldots\ \left((x =)\ 0.987y - 3.22\right)$ ***A1***

$(x =)\ 0.987124\ldots(90) - 3.21970\ldots\ (= 85.6214...)$ ***A1***

$= 86$ to nearest integer. ***AG***

Note: Condone notation for x and y switched if values are correct.

[4 marks]

Total [7 marks]

6. let X be the number of days of rain in May

(a) recognition of binomial distribution ***(M1)***

$X \sim \mathrm{B}(31, 0.2)$ or ${}^{31}C_{10}(0.2)^{10}(0.8)^{21}$ or $X \sim \mathrm{B}(n, p)$ or ${}^{n}C_{r}p^{r}(1-p)^{n-r}$

$\mathrm{P}(X = 10) = 0.0418894\ldots$

$= 0.0419$ ***A1***

Note: If no working shown, award ***(M1)A0*** for 0.042 (2 sf)

[2 marks]

(b) recognition of need to find $\mathrm{P}(X \geq 10)\left(= 1 - \mathrm{P}(X \leq 9)\right)$ ***(M1)***

$= 0.0745998\ldots.(= 1 - 0.925400\ldots)$

$= 0.0746$ ***A1***

Note: If no working shown, award ***(M1)A0*** for 0.075 (2 sf)

[2 marks]

(c) recognition of 9 days with no rain followed by a day of rain ***(M1)***

$0.8^{9} \times 0.2 = 0.0268435\ldots$

$= 0.0268$ ***A1***

Note: If no working shown, award ***(M1)A0*** for 0.027 (2 sf)

[2 marks]

Total [6 marks]

7. $\frac{dy}{dx} - y = x$

recognition that an integrating factor is required ***(M1)***

$e^{\int P(x)dx} \left(= e^{\int -1 dx}\right)$

$= e^{-x}$ ***(A1)***

$e^{-x}\frac{dy}{dx} - e^{-x}y = xe^{-x}$

$e^{-x}y\left(= \int xe^{-x}dx\right)$ ***A1***

attempt to integrate right hand side using integration by parts ***(M1)***

$\left(e^{-x}y =\right) - xe^{-x} + \int e^{-x}dx$

$= -xe^{-x} - e^{-x} + c$ ***A1***

$y = -x - 1 + ce^{x}$

substitute initial values $y = 2$ and $x = 0$ into an integrated expression involving c ***(M1)***

$2 = -1 + c \Rightarrow c = 3$

$y = 3e^{x} - x - 1$ ***A1***

[7 marks]

8. (a) $\int_0^k kx\,\mathrm{d}x + \int_k^{2k} \left(2kx - x^2\right)\mathrm{d}x = 1$ ***A1***

$\left[\frac{kx^2}{2}\right]_0^k + \left[kx^2 - \frac{x^3}{3}\right]_k^{2k} = 1$

$\frac{k^3}{2} + \left(4k^3 - \frac{8k^3}{3}\right) - \left(k^3 - \frac{k^3}{3}\right) = 1$ ***A1***

$7k^3 = 6$ ***AG***

[2 marks]

(b) recognition that the median m is the value such that $\int_0^m f\left(x\right)\mathrm{d}x = 0.5$ or $\int_m^{2k} f\left(x\right)\mathrm{d}x = 0.5$ ***(M1)***

($k = 0.949914\ldots\left(= \sqrt[3]{\frac{6}{7}}\right)$ so) $\int_0^k kx\,\mathrm{d}x = 0.428571\ldots\left(= \frac{3}{7}\right)$ seen anywhere ***(A1)***

Note: This ***A1*** is independent of ***M1***.

EITHER

($m > k$ so) $\int_0^k kx\,\mathrm{d}x + \int_k^m \left(2kx - x^2\right)\mathrm{d}x = 0.5$ OR $\int_k^m \left(2kx - x^2\right)\mathrm{d}x = 0.0714285\ldots\left(= \frac{1}{14}\right)$ ***(A1)***

$\left[kx^2 - \frac{x^3}{3}\right]_k^m = 0.0714285\ldots$

$m = 1.02925\ldots$

continued…

Question 8 continued

OR

($m > k$ so) $\int_m^{2k} \left(2kx - x^2\right) \mathrm{d}x = 0.5$ ***(A1)***

$$\left[kx^2 - \frac{x^3}{3}\right]_m^{2k} = 0.5$$

$m = 1.02925\ldots$

THEN

$m = 1.03$ ***A1***

Note: The correct 3sf answer can be found by solving $\int_0^m kx\mathrm{d}x = 0.5$. This method is not valid since $m > k$. In this case award ***M1A0M0A0.***

[4 marks]

Total [6 marks]

9. (a) **METHOD 1**

the number of ways Alvin and Bobby can be seated is $2\times 8(=16)$ ***(A1)***

the number of ways the other children can be seated is $8!(=40320)$ ***(A1)***

Note: These ***A1*** marks may be awarded independently.

total number of ways is $(16\times 8! =)645120$ ***A1***

Note: Accept $16\times 8!$ and 645000 .

METHOD 2

the number of ways children can be seated in a row of 10 seats is $2\times 9!(=725760)$ ***(A1)***

the number of ways the children can be seated with Alvin and Bobby in seats 5 and 6 is $2\times 8!(=80640)$ ***(A1)***

Note: These ***A1*** marks may be awarded independently.

total number of ways is $(2\times 9! - 2\times 8! =)645120$ ***A1***

Note: Accept $16\times 8!$ and 645000 .

[3 marks]

(b) **METHOD 1**

attempt to find number of ways that A and B are seated next to each other AND C and D are seated next to each other and subtract from part a) ***(M1)***

<u>Case 1</u>: A and B are sat at the end of a row (8 ways)

$6(2)=12$ ways to seat C and D together

$12\times 6!(=8640)$ ways ***(A1)***

the total number of ways is $8\times 12\times 6!(=69120)$

continued…

Question 9 continued

<u>Case 2</u>: A and B are not sat at the end of a row (8 ways)

$5(2)=10$ ways to seat C and D together

$10\times 6!=7200$ ways **(A1)**

the total number of ways is $8\times 10\times 6!\left(=57600\right)$

total number of ways is $645120-\left(69120+57600\right)$

$=518400$ **A1**

Note: Accept 518000 or 518280 (from use of 645000).

METHOD 2

attempt to split into cases based on position of A and B and adding all possibilities **(M1)**

<u>Case 1</u>: A and B are sat at the end of a row (8 ways)

the number of ways C and D can be seated:

with at least one in the same row as A and B $2(1+15)=32$

with both in a different row to A and B $2(6)=12$

the number of ways C and D can be seated is $44\times 6!\left(=31680\right)$ **(A1)**

the total number of ways is $8\times 31680=253440$

<u>Case 2</u>: A and B are not sat at the end of a row (8 ways)

the number of ways C and D can be seated:

with at least one in the same row as A and B $2(2+15)=34$

with both in a different row to A and B $2(6)=12$

the number of ways C and D can be seated is $46\times 6!\left(=33120\right)$ **(A1)**

the total number of ways is $8\times 33120=264960$

total number of ways is $253440+264960$

$=518400$ **A1**

Note: Accept 518000.

[4 marks]

Total [7 marks]

Section B

10. (a) **EITHER**

attempt to find value of t for the first low tide OR the first high tide ***(M1)***

$11.2619\ldots - 5.13801\ldots$

$= 6.12396\ldots$ ***(A1)***

OR

attempt to find half of the period ***(M1)***

$\frac{1}{2} \times \frac{2\pi}{0.513}$

$= 6.12396\ldots$ ***(A1)***

THEN

$m = (6.12396\ldots - 6) \times 60 = 7.43773...$

$m = 7$ ***A1***

[3 marks]

(b) attempt to solve $H(t) = 1$ ***(M1)***

$3.56919\ldots$ OR $6.70684\ldots$ OR $15.8171\ldots$ OR $18.9547\ldots$

$(6.70684\ldots - 3.56919\ldots =)\ 3.13764...$

$= 3.14$ (hours) ***A1***

[2 marks]

(c) recognition that $H'(13)$ is required ***(M1)***

$= -0.650622\ldots$

$= -0.651$ (m/h) ***A1***

[2 marks]

continued…

Question 10 continued

(d)

Note: In part (d), award the marks for a, b, c and d independent of each other.

METHOD 1

$a = 1.17$ ***A1***

$d = 1.57$ ***A1***

attempt to find time between low and high tide in hours ***(M1)***

6 hours and 21 minutes = 6.35 hours

(period =) 12.7 ***(A1)***

$$b = \frac{2\pi}{12.7} = 0.494739\ldots$$

$b = 0.495 \left(= \frac{60\pi}{381}\right)$ ***A1***

attempt to find mean of low and high tide times OR substitute values of a known point ***(M1)***

$c = \frac{1}{2}\left(2\frac{41}{60} + 9\frac{2}{60}\right)$ OR eg $0.40 = 1.17\sin\left(0.495\left(2.68333... - c\right)\right) + 1.57$

$c = 5.85833...$

$c = 5.86$ ***A1***

Note: Award ***(M1)A1*** for $c = 18.6$. Award ***(M1)A0*** for $c = -6.84$.

[7 marks]

continued…

Question 10 continued

METHOD 2

$a = 1.17$ **A1**

$d = 1.57$ **A1**

substituting at least one point into $h(t)$ ***(M1)***

$1.17\sin\left(b\left(2\frac{41}{60} - c\right)\right) + 1.57 = 0.4$ OR $1.17\sin\left(b\left(9\frac{2}{60} - c\right)\right) + 1.57 = 2.74$

$b\left(2\frac{41}{60} - c\right) = -\frac{\pi}{2}(= -1.57)$ **AND** $b\left(9\frac{2}{60} - c\right) = \frac{\pi}{2}(= 1.57)$ ***(A1)***

Note:	accept any angles of the form $-\frac{\pi}{2} + c\pi k$ and $\frac{\pi}{2} + c\pi k$.

EITHER

use of graph or table to find their intersection ***(M1)***

OR

attempt to solve their equations simultaneously ***(M1)***

$$\frac{2\frac{41}{60} - c}{9\frac{2}{60} - c} = -1$$

THEN

$c = 5.85833...$

$c = 5.86$ ***A1***

$b = 0.494739...$

$b = 0.495$ ***A1***

[7 marks]

(e) attempt to find point of intersection of two graphs ***(M1)***

$T = 4.16292\ldots$ OR $T = 4.16417\ldots$(using 3 sf)

$T = 4.16$ ***A1***

[2 marks]

Total [16 marks]

11. (a) attempt to use implicit differentiation ***M1***

$$\left(1+\frac{\mathrm{d}y}{\mathrm{d}x}\right)\mathrm{e}^{x+y} = 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$ ***A1A1***

Note: Award ***A1*** for LHS and ***A1*** for RHS.

attempt to expand brackets and collect $\frac{\mathrm{d}y}{\mathrm{d}x}$ terms on the same side ***M1***

$\mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \mathrm{e}^{x+y}$ OR $\mathrm{e}^{x+y}\frac{\mathrm{d}y}{\mathrm{d}x} - 2y\frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^{x+y} = 2x$ OR $\left(\mathrm{e}^{x+y} - 2y\right)\frac{\mathrm{d}y}{\mathrm{d}x} = 2x - \mathrm{e}^{x+y}$ ***A1***

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - \mathrm{e}^{x+y}}{\mathrm{e}^{x+y} - 2y}$$ ***AG***

[5 marks]

(b) (i) recognition that $2x - \mathrm{e}^{x+y} = 0$ at P and Q ***(M1)***

$\Rightarrow x + y = \ln(2x)$ ***A1***

recognition that $\mathrm{e}^{x+y} = x^2 + y^2$ at P and Q ***(M1)***

$\Rightarrow 2x = x^2 + \left(\ln(2x) - x\right)^2$ or equivalent ***A1***

$2x^2 + \left(\ln(2x)\right)^2 - 2x\ln(2x) - 2x = 0$ ***AG***

continued…

Question 11 continued

(ii) $x = 0.331077\ldots$ or $1.84273\ldots$

$x = 0.331$ or 1.84 ***A1A1***

attempt to use $x + y = \ln(2x)$ OR $e^{x+y} = x^2 + y^2$ to find y-coordinates ***(M1)***

$y = -0.743332\ldots,\ y = -0.538335...$

coordinates $(0.331, -0.743)$ and $(1.84, -0.538)$ ***A1A1***

Note: If no working shown, award ***A1A1(M1)A0A0*** for $(0.33, -0.74)$ and $(1.8, -0.54)$ (2 sf)

[9 marks]

(c) coordinates $(-0.743, 0.331)$ and $(-0.538, 1.84)$ ***A1***

Note: Do not award ***FT*** from (b) if only one coordinate pair is given.

[1 mark]

(d) setting $\dfrac{2x - e^{x+y}}{e^{x+y} - 2y} = -1$ OR recognition that the point lies on line of symmetry ***(M1)***

$2x - e^{x+y} = 2y - e^{x+y}$

$y = x$ ***(A1)***

attempt to substitute $y = x$ into $e^{x+y} = x^2 + y^2$ ***(M1)***

$e^{2x} = 2x^2$ OR $e^{2y} = 2y^2$

$x = -0.451,\ y = -0.451$ ***A1***

coordinates $(-0.451, -0.451)$

Note: If no working shown, award ***M1A1(M1)A0*** for $x = -0.45,\ y = -0.45$ (2 sf)

[4 marks]

Total [19 marks]

12. (a) Let θ be the angle between $\boldsymbol{u}$ and $\boldsymbol{v}$.

$(\boldsymbol{u}\cdot\boldsymbol{v})^2+|\boldsymbol{u}\times\boldsymbol{v}|^2$

$=(|\boldsymbol{u}||\boldsymbol{v}|\cos\theta)^2+(|\boldsymbol{u}||\boldsymbol{v}|\sin\theta)^2$ OR $|\boldsymbol{u}|^2|\boldsymbol{v}|^2\cos^2\theta+|\boldsymbol{u}|^2|\boldsymbol{v}|^2\sin^2\theta$ ***(A1)***

$=|\boldsymbol{u}|^2|\boldsymbol{v}|^2(\cos^2\theta+\sin^2\theta)$ ***A1***

$=|\boldsymbol{u}|^2|\boldsymbol{v}|^2$ ***AG***

[2 marks]

(b) (i) $|\boldsymbol{u}\times\boldsymbol{v}|=2\sqrt{6}\,(=4.89897...=4.90)$ ***A1***

(ii) $\boldsymbol{v}=\begin{pmatrix}3\\1\\-1\end{pmatrix}$ ***(A1)***

$|\boldsymbol{v}|=\sqrt{3^2+1^2+(-1)^2}\ \left(=\sqrt{11}=3.31662\ldots\right)$ ***(A1)***

substitution of values $\boldsymbol{u}\cdot\boldsymbol{v}, |\boldsymbol{u}\times\boldsymbol{v}|$ and $|\boldsymbol{v}|$ into $(\boldsymbol{u}\cdot\boldsymbol{v})^2+|\boldsymbol{u}\times\boldsymbol{v}|^2=|\boldsymbol{u}|^2|\boldsymbol{v}|^2$ ***M1***

$3^2+\left(2\sqrt{6}\right)^2=|\boldsymbol{u}|^2\ \left|\sqrt{11}\right|^2$

$|\boldsymbol{u}|=\sqrt{3}\,(=1.73205...=1.73)$ ***A1***

continued…

Question 12 continued

(iii) $\boldsymbol{u} = \begin{pmatrix} p \\ q-1 \\ 1 \end{pmatrix}$ ***(A1)***

attempt to use $\boldsymbol{u} \cdot \boldsymbol{v} = 3$ ***(M1)***

$\begin{pmatrix} p \\ q-1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} = 3$

$3p + q = 5 \left(\Rightarrow q = 5 - 3p \right)$ ***A1***

attempt to use $|\boldsymbol{u}| = \sqrt{3}$ ***(M1)***

$p^2 + (q-1)^2 + 1^2 = \sqrt{3}^2 \left(\Rightarrow p^2 + q^2 - 2q + 1 + 1 = 3 \right)$

Note: Award ***M1*** for use of $|\boldsymbol{u} \times \boldsymbol{v}|^2 = q^2 + (p+3)^2 + (p - 3q + 3)^2 \left(= \left(2\sqrt{6}\right)^2 \right)$.

attempt to form quadratic in one variable, p or q ***(M1)***

$p^2 + (4-3p)^2 = 2$ OR $p^2 + (5-3p)^2 - 2p = 1$ OR $10p^2 - 24p + 14 = 0$ OR

$\left(\frac{5-q}{3} \right)^2 + (q-1)^2 = 2$ OR $\left(\frac{5-q}{3} \right)^2 + q^2 - 2q = 1$ OR $10q^2 - 28q + 16 = 0$ ***(A1)***

$p = 1$ or $p = 1.4 \left(= \frac{7}{5} \right)$ ***A1***

$q = 2$ or $q = 0.8 \left(= \frac{4}{5} \right)$ ***A1***

Note: Award final ***A1*** marks for correct values, even if the p values and q values are not explicitly paired.

[13 marks]

continued…

Question 12 continued

(c) **METHOD 1**

attempt to express $\boldsymbol{w} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in terms of one variable ***(M1)***

$\boldsymbol{v} \cdot \boldsymbol{w} = 3x + y - z = 0,\ \boldsymbol{u} \cdot \boldsymbol{w} = x + y + z = 0$

$y = -2x$ and $z = x$ OR $x = z = -\frac{1}{2}y$ OR $x = z$ and $y = -2z$ ***(A1)***

attempt to use area of a triangle = $\frac{\text{base} \times \text{height}}{2} = \frac{|\boldsymbol{w}||\boldsymbol{v}|}{2}$ ***(M1)***

$\frac{\sqrt{x^2 + y^2 + z^2}\sqrt{11}}{2} = 5$

$6x^2 = \frac{100}{11}$ or $\frac{3y^2}{2} = \frac{100}{11}$ or $6z^2 = \frac{100}{11}$

$x = \pm 1.2309\ldots\left(= \pm\frac{5\sqrt{66}}{33}\right)$ OR $y = \mp 2.4618\ldots\left(\mp\frac{10\sqrt{66}}{33}\right)$ OR

$z = \pm 1.2309\ldots\left(= \pm\frac{5\sqrt{66}}{33}\right)$ ***A1***

$\boldsymbol{w} = \pm\begin{pmatrix} 1.23091\ldots \\ -2.46182\ldots \\ 1.23091\ldots \end{pmatrix}$

$\boldsymbol{w} = \pm\begin{pmatrix} 1.23 \\ -2.46 \\ 1.23 \end{pmatrix}$ or $\boldsymbol{w} = \pm\frac{5}{\sqrt{66}}\begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}$ ***A1***

Note: If no working shown, award ***M1A1(M1)A1A0*** for $\boldsymbol{w} = \pm\begin{pmatrix} 1.2 \\ -2.5 \\ 1.2 \end{pmatrix}$ (2 sf)

continued…

Question 12 continued

METHOD 2

attempt to write $\boldsymbol{u}\times\boldsymbol{v}$ as a multiple of $\boldsymbol{w}$ or recognizing that $\boldsymbol{w}$ is normal to $\boldsymbol{u}$ and $\boldsymbol{v}$ ***(M1)***

$$\boldsymbol{w}=\boldsymbol{u}\times\boldsymbol{v}=\lambda\begin{pmatrix}-2\\4\\-2\end{pmatrix}\left(=\begin{pmatrix}-2\lambda\\4\lambda\\-2\lambda\end{pmatrix}\right)$$ ***(A1)***

attempt to use area of a triangle = $\dfrac{\text{base}\times\text{height}}{2}$ OR area = $\dfrac{1}{2}\left|\boldsymbol{v}\times\boldsymbol{w}\right|$ ***(M1)***

$$\frac{\sqrt{(-2\lambda)^2+(4\lambda)^2+(-2\lambda)^2}\sqrt{11}}{2}=5 \text{ OR } \frac{1}{2}\left|\boldsymbol{v}\times\boldsymbol{w}\right|=\lambda\begin{vmatrix}1\\4\\7\end{vmatrix}=5$$

$$(-2\lambda)^2+(4\lambda)^2+(-2\lambda)^2=\frac{100}{11} \text{ OR } \lambda^2+(4\lambda)^2+(7\lambda)^2=25$$

$$24\lambda^2=\frac{100}{11} \text{ OR } 66\lambda^2=25$$

$$\lambda=\pm0.61545\ldots\left(=\pm\frac{5\sqrt{66}}{66}\right)$$ ***A1***

$$\boldsymbol{w}=\pm\begin{pmatrix}1.23091\ldots\\-2.46182\ldots\\1.23091\ldots\end{pmatrix}$$

$$\boldsymbol{w}=\pm\begin{pmatrix}1.23\\-2.46\\1.23\end{pmatrix} \text{ or } \boldsymbol{w}=\pm\frac{5}{\sqrt{66}}\begin{pmatrix}-2\\4\\-2\end{pmatrix}$$ ***A1***

Note: If no working shown, award ***M1A1(M1)A1A0*** for $\boldsymbol{w}=\pm\begin{pmatrix}1.2\\-2.5\\1.2\end{pmatrix}$ (2 sf)

continued…

Question 12 continued

METHOD 3

recognising $\frac{1}{2}|\boldsymbol{u}\times\boldsymbol{w}| = 5\times\frac{|\boldsymbol{u}|}{|\boldsymbol{v}|}\ \left(=\frac{5\sqrt{3}}{\sqrt{11}}\right)$ ***(M1)***

since $\boldsymbol{w}$ is perpendicular to both $\boldsymbol{u}$ and $\boldsymbol{v}$, $\boldsymbol{w}$ is a mulitple of their normal

$$\Rightarrow \boldsymbol{w} = \lambda\begin{pmatrix}-2\\4\\-2\end{pmatrix}$$ ***(A1)***

attempt to find $\boldsymbol{u}\times\boldsymbol{w}$ ***(M1)***

$$\boldsymbol{u}\times\boldsymbol{w} = \begin{pmatrix}1\\1\\1\end{pmatrix}\times\lambda\begin{pmatrix}-2\\4\\-2\end{pmatrix} = \lambda\begin{pmatrix}-6\\0\\6\end{pmatrix}$$

$$|\boldsymbol{u}\times\boldsymbol{w}| = \lambda\sqrt{72}$$

$$\frac{1}{2}\lambda\sqrt{72} = 5\frac{\sqrt{3}}{\sqrt{11}} \Rightarrow \lambda = \frac{10}{\sqrt{264}}\ \ (=0.615457...)$$ ***A1***

$$\boldsymbol{w} = \pm\begin{pmatrix}1.23091\ldots\\-2.46182\ldots\\1.23091\ldots\end{pmatrix}$$

$\boldsymbol{w} = \pm\begin{pmatrix}1.23\\-2.46\\1.23\end{pmatrix}$ or $\boldsymbol{w} = \pm\frac{5}{\sqrt{66}}\begin{pmatrix}-2\\4\\-2\end{pmatrix}$ ***A1***

Note: If no working shown, award ***M1A1(M1)A1A0*** for $\boldsymbol{w} = \pm\begin{pmatrix}1.2\\-2.5\\1.2\end{pmatrix}$ (2 sf)

[5 marks]

Total [20 marks]