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**Mathematics: analysis and approaches**  
**Higher level**  
**Paper 3**

Tuesday 11 May 2021 (morning)

1 hour

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**Instructions to candidates**

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

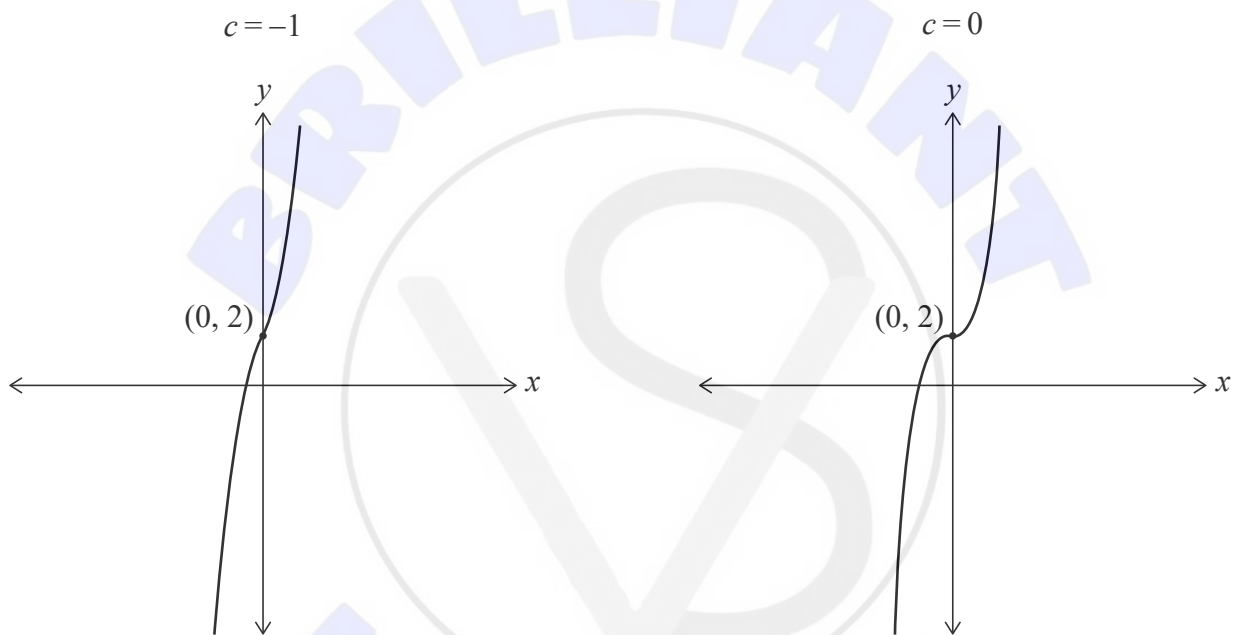
Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 27]

**This question asks you to explore the behaviour and key features of cubic polynomials of the form  $x^3 - 3cx + d$ .**

Consider the function  $f(x) = x^3 - 3cx + 2$  for  $x \in \mathbb{R}$  and where  $c$  is a parameter,  $c \in \mathbb{R}$ .

The graphs of  $y = f(x)$  for  $c = -1$  and  $c = 0$  are shown in the following diagrams.



(a) On separate axes, sketch the graph of  $y = f(x)$  showing the value of the  $y$ -intercept and the coordinates of any points with zero gradient, for

(i)  $c = 1$ ; [3]

(ii)  $c = 2$ . [3]

(b) Write down an expression for  $f'(x)$ . [1]

**(This question continues on the following page)**

**(Question 1 continued)**

- (c) Hence, or otherwise, find the set of values of  $c$  such that the graph of  $y = f(x)$  has
- (i) a point of inflexion with zero gradient; [1]
  - (ii) one local maximum point and one local minimum point; [2]
  - (iii) no points where the gradient is equal to zero. [1]
- (d) Given that the graph of  $y = f(x)$  has one local maximum point and one local minimum point, show that
- (i) the  $y$ -coordinate of the local maximum point is  $2c^{\frac{3}{2}} + 2$ ; [3]
  - (ii) the  $y$ -coordinate of the local minimum point is  $-2c^{\frac{3}{2}} + 2$ . [1]
- (e) Hence, for  $c > 0$ , find the set of values of  $c$  such that the graph of  $y = f(x)$  has
- (i) exactly one  $x$ -axis intercept; [2]
  - (ii) exactly two  $x$ -axis intercepts; [2]
  - (iii) exactly three  $x$ -axis intercepts. [2]

Consider the function  $g(x) = x^3 - 3cx + d$  for  $x \in \mathbb{R}$  and where  $c, d \in \mathbb{R}$ .

- (f) Find all conditions on  $c$  and  $d$  such that the graph of  $y = g(x)$  has exactly one  $x$ -axis intercept, explaining your reasoning. [6]

Turn over

2. [Maximum mark: 28]

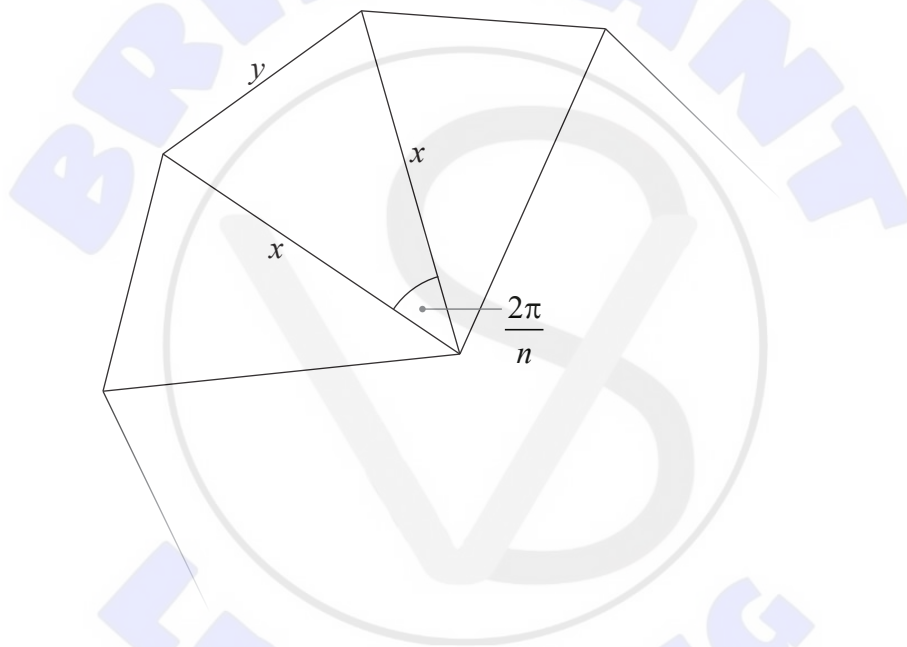
**This question asks you to examine various polygons for which the numerical value of the area is the same as the numerical value of the perimeter. For example, a 3 by 6 rectangle has an area of 18 and a perimeter of 18.**

For each polygon in this question, let the numerical value of its area be  $A$  and let the numerical value of its perimeter be  $P$ .

(a) Find the side length,  $s$ , where  $s > 0$ , of a square such that  $A = P$ . [3]

An  $n$ -sided regular polygon can be divided into  $n$  congruent isosceles triangles. Let  $x$  be the length of each of the two equal sides of one such isosceles triangle and let  $y$  be the length of the third side. The included angle between the two equal sides has magnitude  $\frac{2\pi}{n}$ .

Part of such an  $n$ -sided regular polygon is shown in the following diagram.



(b) Write down, in terms of  $x$  and  $n$ , an expression for the area,  $A_T$ , of one of these isosceles triangles. [1]

(c) Show that  $y = 2x \sin \frac{\pi}{n}$ . [2]

Consider a  $n$ -sided regular polygon such that  $A = P$ .

(d) Use the results from parts (b) and (c) to show that  $A = P = 4n \tan \frac{\pi}{n}$ . [7]

**(This question continues on the following page)**

**(Question 2 continued)**

The Maclaurin series for  $\tan x$  is  $x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$

- (e) (i) Use the Maclaurin series for  $\tan x$  to find  $\lim_{n \rightarrow \infty} \left( 4n \tan \frac{\pi}{n} \right)$ . [3]
- (ii) Interpret your answer to part (e)(i) geometrically. [1]

Consider a right-angled triangle with side lengths  $a$ ,  $b$  and  $\sqrt{a^2 + b^2}$ , where  $a \geq b$ , such that  $A = P$ .

- (f) Show that  $a = \frac{8}{b-4} + 4$ . [7]
  - (g) (i) By using the result of part (f) or otherwise, determine the three side lengths of the only two right-angled triangles for which  $a, b, A, P \in \mathbb{Z}$ . [3]
  - (ii) Determine the area and perimeter of these two right-angled triangles. [1]
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**References:**

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