

© International Baccalaureate Organization 2023

All rights reserved. No part of this product may be reproduced in any form or by any electronic or mechanical means, including information storage and retrieval systems, without the prior written permission from the IB. Additionally, the license tied with this product prohibits use of any selected files or extracts from this product. Use by third parties, including but not limited to publishers, private teachers, tutoring or study services, preparatory schools, vendors operating curriculum mapping services or teacher resource digital platforms and app developers, whether fee-covered or not, is prohibited and is a criminal offense.

More information on how to request written permission in the form of a license can be obtained from <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

© Organisation du Baccalauréat International 2023

Tous droits réservés. Aucune partie de ce produit ne peut être reproduite sous quelque forme ni par quelque moyen que ce soit, électronique ou mécanique, y compris des systèmes de stockage et de récupération d'informations, sans l'autorisation écrite préalable de l'IB. De plus, la licence associée à ce produit interdit toute utilisation de tout fichier ou extrait sélectionné dans ce produit. L'utilisation par des tiers, y compris, sans toutefois s'y limiter, des éditeurs, des professeurs particuliers, des services de tutorat ou d'aide aux études, des établissements de préparation à l'enseignement supérieur, des fournisseurs de services de planification des programmes d'études, des gestionnaires de plateformes pédagogiques en ligne, et des développeurs d'applications, moyennant paiement ou non, est interdite et constitue une infraction pénale.

Pour plus d'informations sur la procédure à suivre pour obtenir une autorisation écrite sous la forme d'une licence, rendez-vous à l'adresse <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

© Organización del Bachillerato Internacional, 2023

Todos los derechos reservados. No se podrá reproducir ninguna parte de este producto de ninguna forma ni por ningún medio electrónico o mecánico, incluidos los sistemas de almacenamiento y recuperación de información, sin la previa autorización por escrito del IB. Además, la licencia vinculada a este producto prohíbe el uso de todo archivo o fragmento seleccionado de este producto. El uso por parte de terceros —lo que incluye, a título enunciativo, editoriales, profesores particulares, servicios de apoyo académico o ayuda para el estudio, colegios preparatorios, desarrolladores de aplicaciones y entidades que presten servicios de planificación curricular u ofrezcan recursos para docentes mediante plataformas digitales—, ya sea incluido en tasas o no, está prohibido y constituye un delito.

En este enlace encontrará más información sobre cómo solicitar una autorización por escrito en forma de licencia: <https://ibo.org/become-an-ib-school/ib-publishing/licensing/applying-for-a-license/>.

Mathematics: analysis and approaches

Higher level

Paper 3

9 May 2023

Zone A afternoon | Zone B morning | Zone C afternoon

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.



Blank page

Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 25]

In this question, you will be investigating the family of functions of the form $f(x) = x^n e^{-x}$.

Consider the family of functions $f_n(x) = x^n e^{-x}$, where $x \geq 0$ and $n \in \mathbb{Z}^+$.

When $n = 1$, the function $f_1(x) = xe^{-x}$, where $x \geq 0$.

(a) Sketch the graph of $y = f_1(x)$, stating the coordinates of the local maximum point. [4]

(b) Show that the area of the region bounded by the graph $y = f_1(x)$, the x -axis and the line $x = b$, where $b > 0$, is given by $\frac{e^b - b - 1}{e^b}$. [6]

You may assume that the total area, A_n , of the region between the graph $y = f_n(x)$ and the x -axis can be written as $A_n = \int_0^\infty f_n(x) dx$ and is given by $\lim_{b \rightarrow \infty} \int_0^b f_n(x) dx$.

(c) (i) Use l'Hôpital's rule to find $\lim_{b \rightarrow \infty} \frac{e^b - b - 1}{e^b}$. You may assume that the condition for applying l'Hôpital's rule has been met. [2]

(ii) Hence write down the value of A_1 . [1]

You are given that $A_2 = 2$ and $A_3 = 6$.

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(i) A_4 ; [2]

(ii) A_5 . [1]

(e) Suggest an expression for A_n in terms of n , where $n \in \mathbb{Z}^+$. [1]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of m , $\lim_{x \rightarrow \infty} x^m e^{-x} = 0$. [8]

Turn over

2. [Maximum mark: 30]

In this question, you will investigate the maximum product of positive real numbers with a given sum.

Consider the two numbers $x_1, x_2 \in \mathbb{R}^+$, such that $x_1 + x_2 = 12$.

(a) Find the product of x_1 and x_2 as a function, f , of x_1 only. [2]

(b) (i) Find the value of x_1 for which the function is maximum. [1]

(ii) Hence show that the maximum product of x_1 and x_2 is 36. [1]

Consider $M_n(S)$ to be the maximum product of n positive real numbers with a sum of S , where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$.

For $n = 2$, the maximum product can be expressed as $M_2(S) = \left(\frac{S}{2}\right)^2$.

(c) Verify that $M_2(S) = \left(\frac{S}{2}\right)^2$ is true for $S = 12$. [1]

Consider n positive real numbers, x_1, x_2, \dots, x_n .

The geometric mean is defined as $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}}$. It is given that the geometric mean is always less than or equal to the arithmetic mean, so $(x_1 \times x_2 \times \dots \times x_n)^{\frac{1}{n}} \leq \frac{(x_1 + x_2 + \dots + x_n)}{n}$.

(d) (i) Show that the geometric mean and arithmetic mean are equal when $x_1 = x_2 = \dots = x_n$. [2]

(ii) Use this result to prove that $M_n(S) = \left(\frac{S}{n}\right)^n$. [4]

(e) Hence determine the value of

(i) $M_3(12)$; [1]

(ii) $M_4(12)$; [1]

(iii) $M_5(12)$. [1]

For $n \in \mathbb{Z}^+$, let $P(S)$ denote the maximum value of $M_n(S)$ across all possible values of n .

(f) Write down the value of $P(12)$ and the value of n at which it occurs. [2]

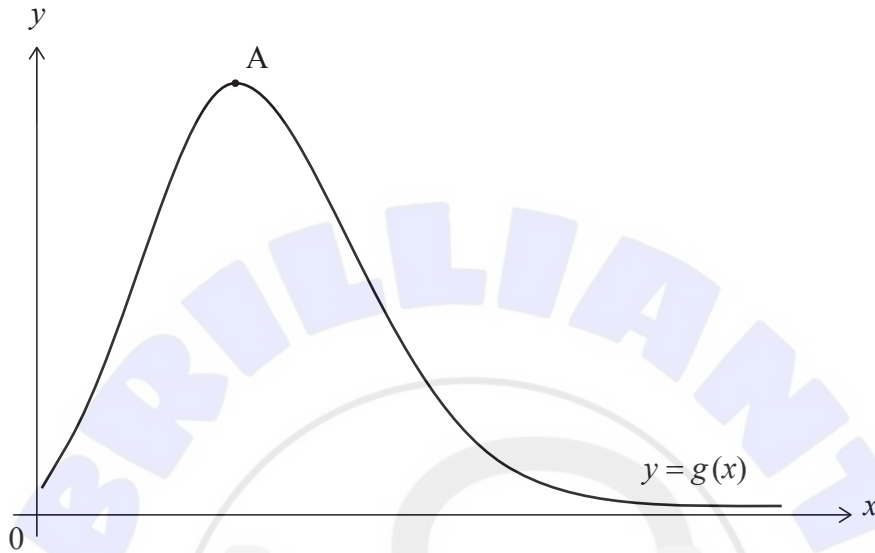
(g) Determine the value of $P(20)$ and the value of n at which it occurs. [3]

(This question continues on the following page)

(Question 2 continued)

Consider the function g , defined by $\ln(g(x)) = x \ln\left(\frac{S}{x}\right)$, where $x \in \mathbb{R}^+$.

A sketch of the graph of $y = g(x)$ is shown in the following diagram. Point A is the maximum point on this graph.



- (h) Find, in terms of S , the x -coordinate of point A. [6]
- (i) Verify that $g(x) = M_x(S)$, when $x \in \mathbb{Z}^+$. [2]
- (j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is 100. Give your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}^+$. [3]

References:

© International Baccalaureate Organization 2023