Diploma Programme
Programme du diplôme
Programa del Diploma

# Mathematics: analysis and approaches
# Higher level
# Paper 3

9 May 2023

**Zone A** afternoon | **Zone B** morning | **Zone C** afternoon

1 hour

### Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[55 marks]**.

5 pages

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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

**1.** [Maximum mark: 25]

**In this question, you will be investigating the family of functions of the form $f(x) = x^n e^{-x}$.**

Consider the family of functions $f_n(x) = x^n e^{-x}$, where $x \geq 0$ and $n \in \mathbb{Z}^+$.

When $n = 1$, the function $f_1(x) = xe^{-x}$, where $x \geq 0$.

(a) Sketch the graph of $y = f_1(x)$, stating the coordinates of the local maximum point. [4]

(b) Show that the area of the region bounded by the graph $y = f_1(x)$, the $x$-axis and the line $x = b$, where $b > 0$, is given by $\dfrac{e^b - b - 1}{e^b}$. [6]

You may assume that the total area, $A_n$, of the region between the graph $y = f_n(x)$ and the $x$-axis can be written as $A_n = \int_0^{\infty} f_n(x)\,dx$ and is given by $\lim_{b\to\infty}\int_0^b f_n(x)\,dx$.

(c) (i) Use l'Hôpital's rule to find $\lim_{b\to\infty}\dfrac{e^b - b - 1}{e^b}$. You may assume that the condition for applying l'Hôpital's rule has been met. [2]

(ii) Hence write down the value of $A_1$. [1]

You are given that $A_2 = 2$ and $A_3 = 6$.

(d) Use your graphic display calculator, and an appropriate value for the upper limit, to determine the value of

(i) $A_4$; [2]

(ii) $A_5$. [1]

(e) Suggest an expression for $A_n$ in terms of $n$, where $n \in \mathbb{Z}^+$. [1]

(f) Use mathematical induction to prove your conjecture from part (e). You may assume that, for any value of $m$, $\lim_{x\to\infty} x^m e^{-x} = 0$. [8]

**2.** [Maximum mark: 30]

**In this question, you will investigate the maximum product of positive real numbers with a given sum.**

Consider the two numbers $x_1$, $x_2 \in \mathbb{R}^+$, such that $x_1 + x_2 = 12$.

(a) Find the product of $x_1$ and $x_2$ as a function, $f$, of $x_1$ only. [2]

(b) (i) Find the value of $x_1$ for which the function is maximum. [1]

(ii) Hence show that the maximum product of $x_1$ and $x_2$ is $36$. [1]

Consider $M_n(S)$ to be the maximum product of $n$ positive real numbers with a sum of $S$, where $n \in \mathbb{Z}^+$ and $S \in \mathbb{R}^+$.

For $n = 2$, the maximum product can be expressed as $M_2(S) = \left(\frac{S}{2}\right)^2$.

(c) Verify that $M_2(S) = \left(\frac{S}{2}\right)^2$ is true for $S = 12$. [1]

Consider $n$ positive real numbers, $x_1, x_2, \ldots, x_n$.

The geometric mean is defined as $(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}}$. It is given that the geometric mean is always less than or equal to the arithmetic mean, so $(x_1 \times x_2 \times \ldots \times x_n)^{\frac{1}{n}} \le \frac{(x_1 + x_2 + \ldots + x_n)}{n}$.

(d) (i) Show that the geometric mean and arithmetic mean are equal when $x_1 = x_2 = \ldots = x_n$. [2]

(ii) Use this result to prove that $M_n(S) = \left(\frac{S}{n}\right)^n$. [4]

(e) Hence determine the value of

(i) $M_3(12)$; [1]

(ii) $M_4(12)$; [1]

(iii) $M_5(12)$. [1]

For $n \in \mathbb{Z}^+$, let $P(S)$ denote the maximum value of $M_n(S)$ across all possible values of $n$.

(f) Write down the value of $P(12)$ and the value of $n$ at which it occurs. [2]

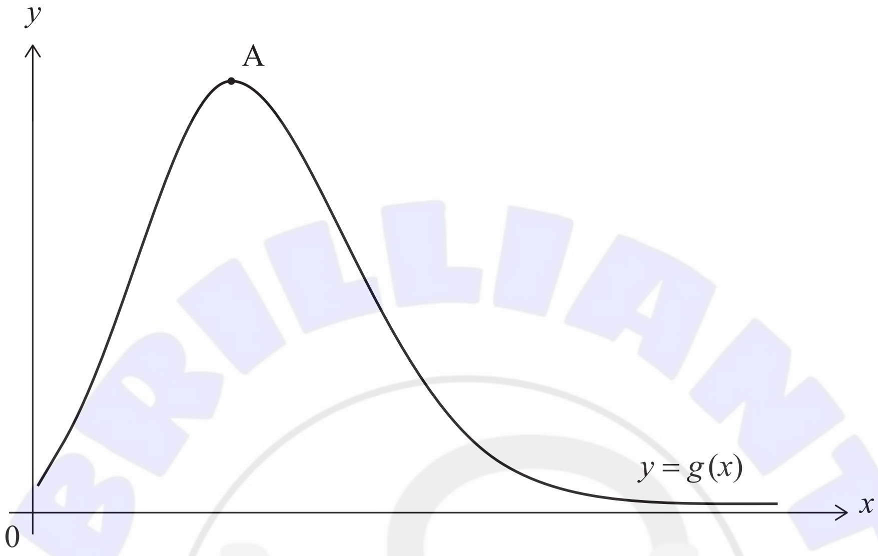
(g) Determine the value of $P(20)$ and the value of $n$ at which it occurs. [3]

**(This question continues on the following page)**

**(Question 2 continued)**

Consider the function $g$, defined by $\ln\left(g(x)\right) = x\ln\left(\frac{S}{x}\right)$, where $x \in \mathbb{R}^+$.

A sketch of the graph of $y = g(x)$ is shown in the following diagram. Point A is the maximum point on this graph.




(h) Find, in terms of $S$, the $x$-coordinate of point A. [6]

(i) Verify that $g(x) = M_x(S)$, when $x \in \mathbb{Z}^+$. [2]

(j) Use your answer to part (h) to find the largest possible product of positive numbers whose sum is $100$. Give your answer in the form $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}^+$. [3]

**References:**