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# Mathematics: applications and interpretation

## Higher level

### Paper 2

9 May 2023

Zone A afternoon | Zone B morning | Zone C afternoon

2 hours

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#### Instructions to candidates

- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Answer all the questions in the answer booklet provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: applications and interpretation formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.



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Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

The mean annual temperatures for Earth, recorded at fifty-year intervals, are shown in the table.

<b>Year (<math>x</math>)</b>	1708	1758	1808	1858	1908	1958	2008
<b>Temperature °C (<math>y</math>)</b>	8.73	9.22	9.10	9.12	9.13	9.45	9.76

Tami creates a linear model for this data by finding the equation of the straight line passing through the points with coordinates (1708, 8.73) and (1958, 9.45).

- (a) Calculate the gradient of the straight line that passes through these two points. [2]
- (b) (i) Interpret the meaning of the gradient in the context of the question.
- (ii) State appropriate units for the gradient. [2]
- (c) Find the equation of this line giving your answer in the form  $y = mx + c$ . [2]
- (d) Use Tami's model to estimate the mean annual temperature in the year 2000. [2]

Thandizo uses linear regression to obtain a model for the data.

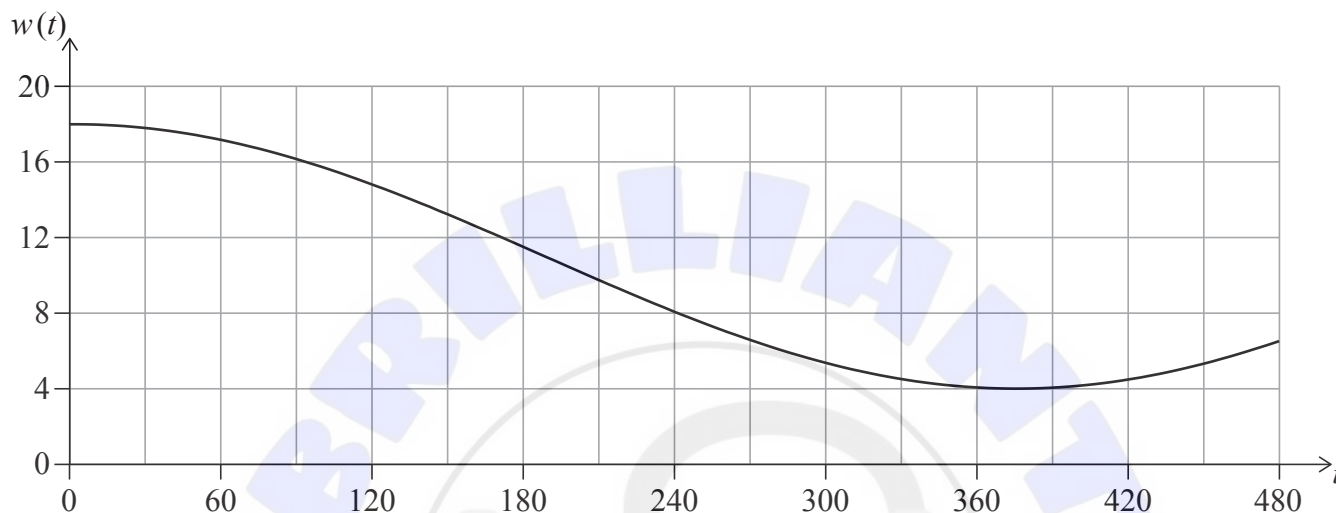
- (e) (i) Find the equation of the regression line  $y$  on  $x$ .
- (ii) Find the value of  $r$ , the Pearson's product-moment correlation coefficient. [3]
- (f) Use Thandizo's model to estimate the mean annual temperature in the year 2000. [2]

**Turn over**

2. [Maximum mark: 15]

The depth of water,  $w$  metres, in a particular harbour can be modelled by the function  $w(t) = a \cos(bt^\circ) + d$  where  $t$  is the length of time, in minutes, after 06:00.

On 20 January, the first high tide occurs at 06:00, at which time the depth of water is 18 m. The following low tide occurs at 12:15 when the depth of water is 4 m. This is shown in the diagram.



- (a) Find the value of  $a$ . [2]
- (b) Find the value of  $d$ . [2]
- (c) Find the period of the function in minutes. [3]
- (d) Find the value of  $b$ . [2]

Naomi is sailing to the harbour on the morning of 20 January. Boats can enter or leave the harbour only when the depth of water is at least 6 m.

- (e) Find the latest time before 12:00, to the nearest minute, that Naomi can enter the harbour. [4]
- (f) Find the length of time (in minutes) between 06:00 and 15:00 on 20 January during which Naomi **cannot** enter or leave the harbour. [2]

3. [Maximum mark: 17]

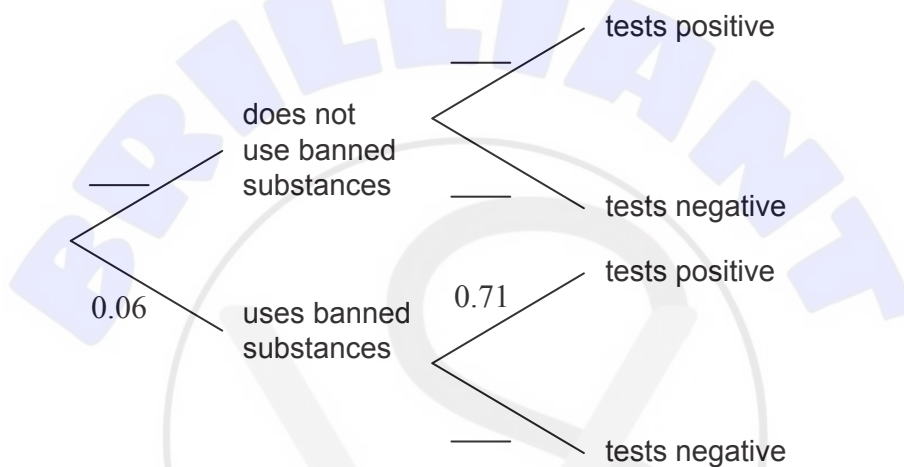
A large international sports tournament tests their athletes for banned substances. They interpret a positive test result as meaning that the athlete uses banned substances. A negative result means that they do not.

The probability that an athlete uses banned substances is estimated to be 0.06.

If an athlete **uses** banned substances, the probability that they will test positive is 0.71.

If an athlete does **not use** banned substances, the probability that they will test negative is 0.98.

- (a) Using the information given, **copy** (into your answer booklet) and complete the following tree diagram. [2]



- (b) (i) Determine the probability that a randomly selected athlete does not use banned substances and tests negative. [4]
- (ii) If two athletes are selected at random, calculate the probability that both athletes do not use banned substances and both test negative. [4]
- (c) (i) Calculate the probability that a randomly selected athlete will receive an **incorrect** test result. [5]
- (ii) A random sample of 1300 athletes at the tournament are selected for testing. Calculate the expected number of athletes in the sample that will receive an incorrect test result. [5]

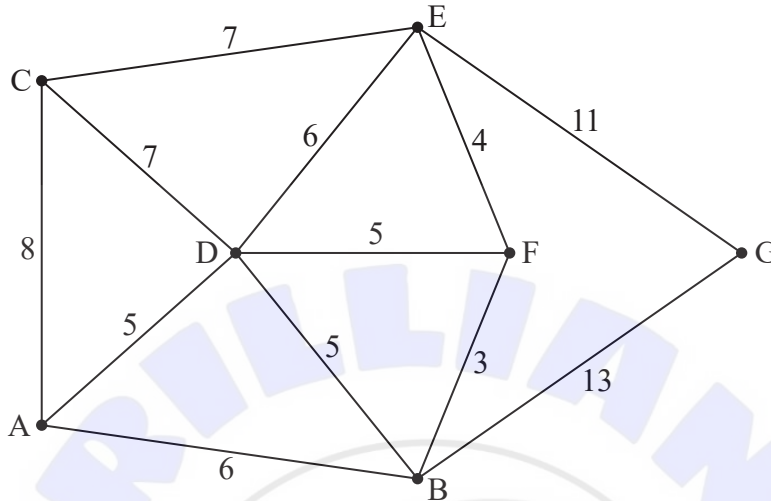
Team X are competing in the tournament. There are 20 athletes in this team. It is known that none of the athletes in Team X use banned substances.

- (d) Calculate the probability that none of the athletes in Team X will test positive. [4]
- (e) Determine the probability that more than 2 athletes in Team X will test positive. [2]

Turn over

4. [Maximum mark: 17]

The vertices in the following graph represent seven towns. The edges represent their connecting roads. The weight on each edge represents the distance, in kilometres, between the two connected towns.



- (a) Determine whether it is possible to complete a journey that starts and finishes at different towns that also uses each of the roads exactly once. Give a reason for your answer.

[2]

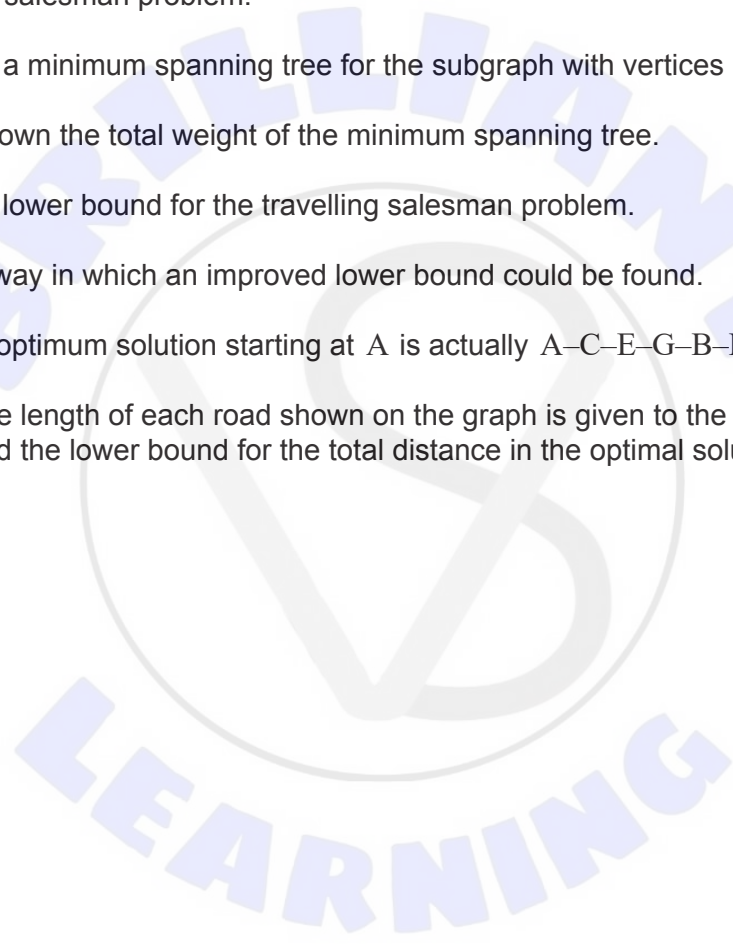
The shortest distance, in kilometres, between any two towns is given in the table.

	A	B	C	D	E	F	G
A	<del>        </del>	6	8	5	11	9	19
B	6	<del>        </del>	12	5	7	3	13
C	8	12	<del>        </del>	7	7	<i>a</i>	<i>b</i>
D	5	5	7	<del>        </del>	6	5	<i>c</i>
E	11	7	7	6	<del>        </del>	4	11
F	9	3	<i>a</i>	5	4	<del>        </del>	<i>d</i>
G	19	13	<i>b</i>	<i>c</i>	11	<i>d</i>	<del>        </del>

(This question continues on the following page)

**(Question 4 continued)**

- (b) Find the value of
- (i)  $a$ ;
  - (ii)  $b$ ;
  - (iii)  $c$ ;
  - (iv)  $d$ . [2]
- (c) Use the nearest neighbour algorithm, starting at vertex  $G$ , to find an upper bound for the travelling salesman problem. [3]
- (d) (i) Sketch a minimum spanning tree for the subgraph with vertices  $A, B, C, D, E, F$ .  
(ii) Write down the total weight of the minimum spanning tree. [4]
- (e) Hence find a lower bound for the travelling salesman problem. [2]
- (f) Explain one way in which an improved lower bound could be found. [1]
- It is found that the optimum solution starting at  $A$  is actually  $A-C-E-G-B-F-D-A$ .
- (g) Given that the length of each road shown on the graph is given to the nearest kilometre, find the lower bound for the total distance in the optimal solution. [3]



5. [Maximum mark: 15]

Goran is interested in the number of sightings of a particular bird each week in the 50 weeks following the first day of September. He collects some data which is shown in the table.

<b>Number of sightings</b>	0	1	2	3	4	5	More than 5
<b>Number of weeks</b>	8	16	13	8	3	2	0

The sample mean number of sightings per week for this data is 1.76.

- (a) Calculate the unbiased estimate of the population variance of sightings per week. [3]

Goran believes that the data follows a Poisson distribution.

- (b) State why your answer to part (a) supports Goran’s belief. [1]

Goran decides to test at the 5% significance level to see if his belief is correct.

His null hypothesis is  $X \sim \text{Po}(m)$ , where the random variable,  $X$ , is defined as the number of sightings per week.

Goran estimates parameter  $m$  to be the mean of the sample, 1.76. He calculates the expected frequencies for sightings per week in the 50 weeks after the first day of September. These are shown to two decimal places in the following table.

<b>Number of sightings</b>	0	1	2	3	4	5 or more
<b>Expected frequencies</b>	8.60	15.14	13.32	7.82	$j$	$k$

- (c) Find the value of
- (i)  $j$ ;
  - (ii)  $k$ . [5]
- (d) State a reason why Goran should combine groups to conduct his significance test. [1]
- (e) Write down the degrees of freedom for the test. [1]
- (f) Find the  $p$ -value for the test. [2]
- (g) State the conclusion of the test. Justify your answer. [2]

6. [Maximum mark: 15]

A model speedboat has its position, at time  $t$  seconds  $t \geq 0$ , defined by

$$\frac{dx}{dt} = 5y - 0.05x, \quad \frac{dy}{dt} = -5x - 0.05y,$$

where  $x$  metres is the distance east and  $y$  metres is the distance north of a fixed point  $O$ .

- (a) Find the eigenvalues of  $A = \begin{pmatrix} -0.05 & 5 \\ -5 & -0.05 \end{pmatrix}$ , giving your answers in the form  $a + bi$ ,

where  $a \neq 0$ ,  $b \neq 0$ .

[4]

- (b) (i) State what  $a \neq 0$  indicates about the path of the speedboat.

- (ii) State what the sign of  $a$  indicates about the path of the speedboat.

[2]

At time  $t = 0$ , the speedboat has position  $(20, 0)$ .

- (c) At time  $t = 0$ , find the value of

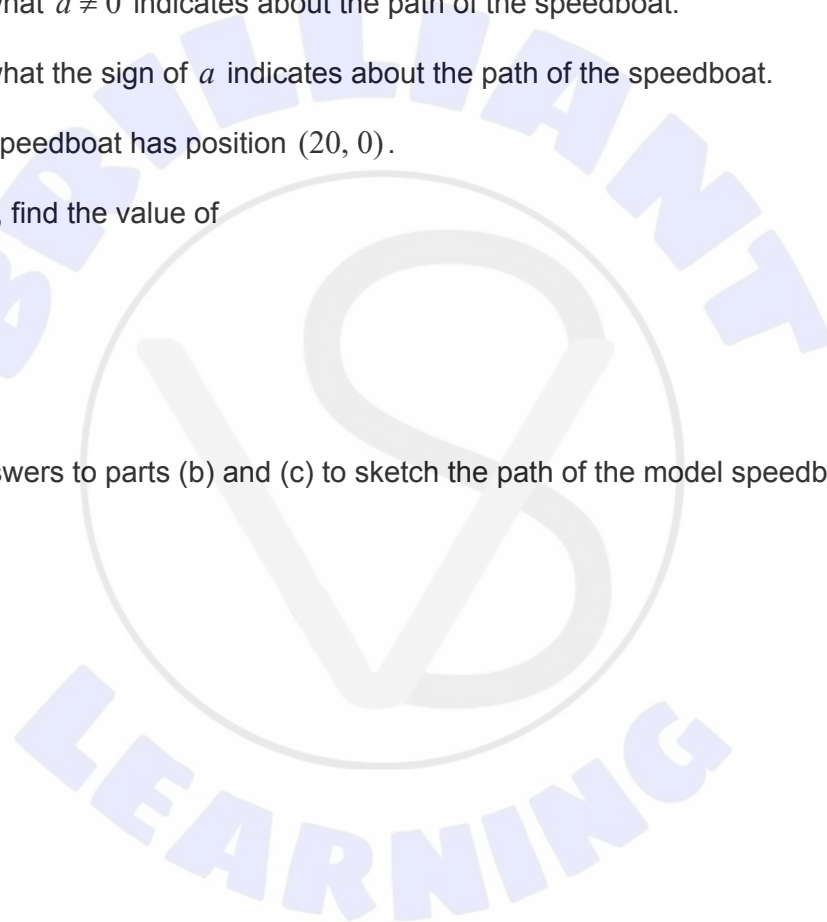
(i)  $\frac{dy}{dt}$ .

(ii)  $\frac{dy}{dx}$ .

[5]

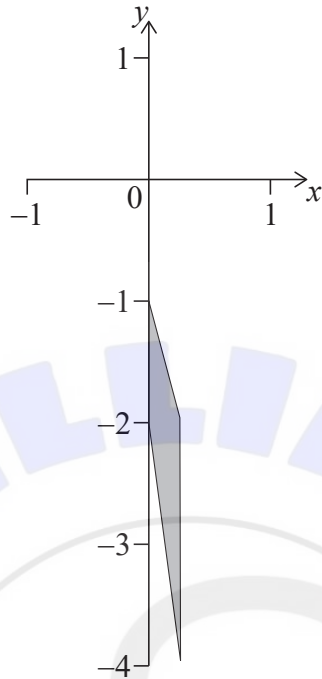
- (d) Use your answers to parts (b) and (c) to sketch the path of the model speedboat.

[4]



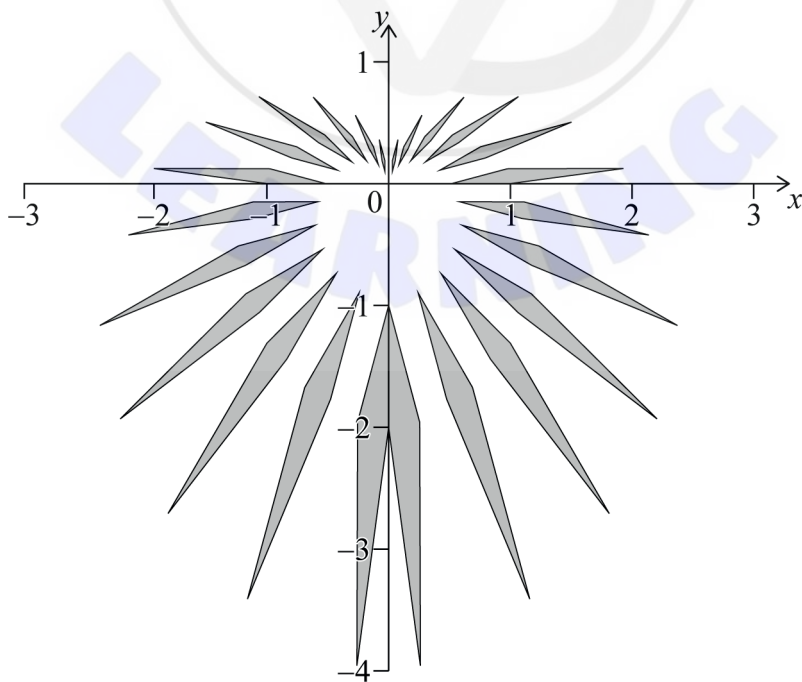
7. [Maximum mark: 18]

A trapezoid,  $Q$ , has vertices  $(0, -1)$ ,  $(0, -2)$ ,  $(\sin 15^\circ, -3 - \cos 15^\circ)$ ,  $(\sin 15^\circ, -1 - \cos 15^\circ)$  as shown.



(a) Show that the area of the trapezoid is  $\frac{3}{2} \sin 15^\circ$ . [2]

A design is created with 24 elements. Each element is obtained by transforming the trapezoid  $Q$ . These elements are shaded in the following diagram such that the  $y$ -axis is a line of symmetry.



(This question continues on the following page)

**(Question 7 continued)**

The transformation that produces each of the elements on the **right side** of the design can be represented by a matrix of the form

$$M_k = \begin{pmatrix} \left(1 - \frac{k}{12}\right) \cos(k \times 15^\circ) & -\left(1 - \frac{k}{12}\right) \sin(k \times 15^\circ) \\ \left(1 - \frac{k}{12}\right) \sin(k \times 15^\circ) & \left(1 - \frac{k}{12}\right) \cos(k \times 15^\circ) \end{pmatrix}$$

where  $k = 0, 1, 2, 3, \dots, 11$ .

- (b) (i) Find the matrix  $M_6$ . Give your answer in the form  $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$  where  $a, b, c, d \in \mathbb{Q}$ .  
 (ii) Hence find the coordinates of the image of the vertex  $(0, -1)$  after it is transformed by the matrix  $M_6$ . [4]

The matrix  $M_k$  can be expressed as the product of a rotation matrix and an enlargement matrix.

- (c) Write down, in terms of  $k$ ,  
 (i) the rotation matrix;  
 (ii) the enlargement matrix;  
 (iii) the angle of the rotation;  
 (iv) the scale factor of the enlargement. [4]
- (d) Using your answer to part (c)(iv), or otherwise, find the determinant of the matrix  $M_k$  in terms of  $k$ . [2]
- (e) Hence, or otherwise, find the total area of the elements in the **whole** design. [4]

Each element on the **left side** of the design can be obtained through a transformation of the trapezoid Q by applying the matrix  $N_k$ , where  $k = 0, 1, 2, 3, \dots, 11$ .

- (f) Write down the matrix  $N_k$  as a product of two matrices. [2]

**References:**