# Mathematics: analysis and approaches
# Higher level
# Paper 1

Monday 1 November 2021 (afternoon)

Candidate session number

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | |

2 hours

## Instructions to candidates

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all questions. Answers must be written within the answer boxes provided.
- Section B: answer all questions in the answer booklet provided. Fill in your session number on the front of the answer booklet, and attach it to this examination paper and your cover sheet using the tag provided.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A clean copy of the **mathematics: analysis and approaches formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[110 marks]**.



14 pages

Please **do not** write on this page.

Answers written on this page will not be marked.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

## Section A

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

**1.** [Maximum mark: 4]

Given that $\frac{dy}{dx} = \cos\left(x - \frac{\pi}{4}\right)$ and $y = 2$ when $x = \frac{3\pi}{4}$, find $y$ in terms of $x$.

**2.** [Maximum mark: 9]

The function $f$ is defined by $f(x)=\dfrac{2x+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$.

(a) Write down the equation of

(i) the vertical asymptote of the graph of $f$;

(ii) the horizontal asymptote of the graph of $f$. [2]

(b) Find the coordinates where the graph of $f$ crosses

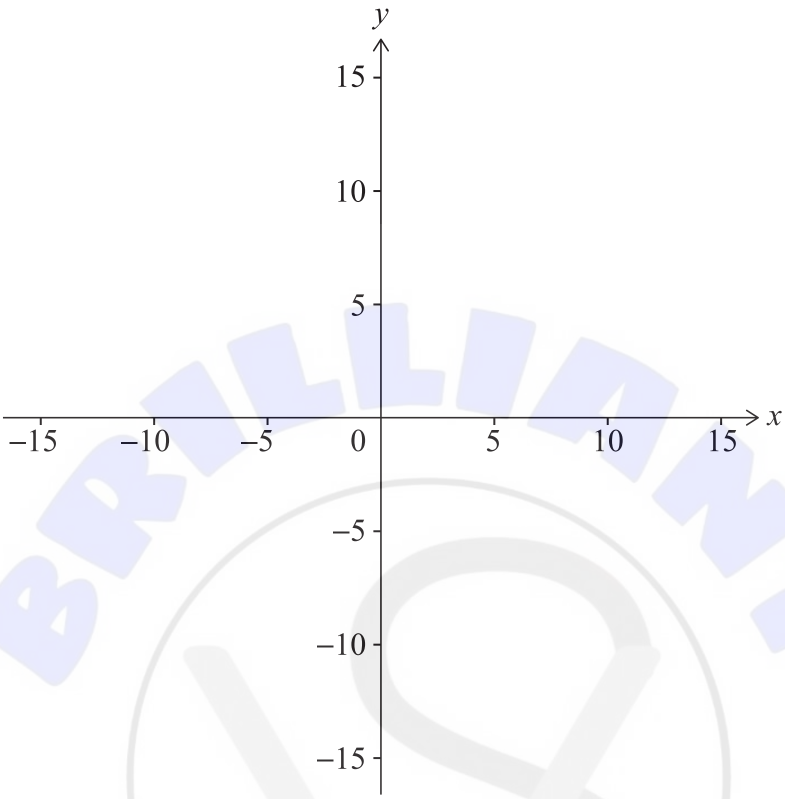
(i) the $x$-axis;

(ii) the $y$-axis. [2]

**(This question continues on the following page)**

**(Question 2 continued)**

(c) Sketch the graph of $f$ on the axes below. [1]




The function $g$ is defined by $g(x) = \dfrac{ax+4}{3-x}$, where $x \in \mathbb{R}$, $x \neq 3$ and $a \in \mathbb{R}$.

(d) Given that $g(x) = g^{-1}(x)$, determine the value of $a$. [4]

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . .

**3.** [Maximum mark: 5]

Solve the equation $\log_3 \sqrt{x} = \dfrac{1}{2\log_2 3} + \log_3\left(4x^3\right)$, where $x > 0$.

**4.** [Maximum mark: 5]

Box 1 contains 5 red balls and 2 white balls.
Box 2 contains 4 red balls and 3 white balls.

(a) A box is chosen at random and a ball is drawn. Find the probability that the ball is red. [3]

Let $A$ be the event that “box 1 is chosen” and let $R$ be the event that “a red ball is drawn”.

(b) Determine whether events $A$ and $R$ are independent. [2]

**5.** [Maximum mark: 7]

The function $f$ is defined for all $x \in \mathbb{R}$. The line with equation $y = 6x - 1$ is the tangent to the graph of $f$ at $x = 4$.

(a) Write down the value of $f'(4)$. [1]

(b) Find $f(4)$. [1]

The function $g$ is defined for all $x \in \mathbb{R}$ where $g(x) = x^2 - 3x$ and $h(x) = f(g(x))$.

(c) Find $h(4)$. [2]

(d) Hence find the equation of the tangent to the graph of $h$ at $x = 4$. [3]

**6.** [Maximum mark: 7]

(a) Show that $2x-3-\dfrac{6}{x-1}=\dfrac{2x^2-5x-3}{x-1}$, $x \in \mathbb{R}$, $x \neq 1$. [2]

(b) Hence or otherwise, solve the equation $2\sin 2\theta-3-\dfrac{6}{\sin 2\theta-1}=0$ for $0 \leq \theta \leq \pi$, $\theta \neq \dfrac{\pi}{4}$. [5]

**7.** [Maximum mark: 7]

The equation $3px^2 + 2px + 1 = p$ has two real, distinct roots.

(a) Find the possible values for $p$. [5]

(b) Consider the case when $p = 4$. The roots of the equation can be expressed in the form $x = \frac{a \pm \sqrt{13}}{6}$, where $a \in \mathbb{Z}$. Find the value of $a$. [2]

**8.** [Maximum mark: 7]

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\ln 2x}{x^2} - \frac{2y}{x}$, $x > 0$, given that $y = 4$ at $x = \frac{1}{2}$.

Give your answer in the form $y = f(x)$.

**9.** [Maximum mark: 7]

Consider the expression $\frac{1}{\sqrt{1+ax}} - \sqrt{1-x}$ where $a \in \mathbb{Q}$, $a \neq 0$.

The binomial expansion of this expression, in ascending powers of $x$, as far as the term in $x^2$ is $4bx + bx^2$, where $b \in \mathbb{Q}$.

(a) Find the value of $a$ and the value of $b$. [6]

(b) State the restriction which must be placed on $x$ for this expansion to be valid. [1]

Do **not** write solutions on this page.

## Section B

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

**10.** [Maximum mark: 16]

A particle $P$ moves along the $x$-axis. The velocity of $P$ is $v\,\text{m}\,\text{s}^{-1}$ at time $t$ seconds, where $v(t) = 4 + 4t - 3t^2$ for $0 \le t \le 3$. When $t = 0$, $P$ is at the origin $\text{O}$.

(a) (i) Find the value of $t$ when $P$ reaches its maximum velocity.

(ii) Show that the distance of $P$ from $\text{O}$ at this time is $\frac{88}{27}$ metres. [7]

(b) Sketch a graph of $v$ against $t$, clearly showing any points of intersection with the axes. [4]

(c) Find the total distance travelled by $P$. [5]

**11.** [Maximum mark: 14]

(a) Prove by mathematical induction that $\frac{\text{d}^n}{\text{d}x^n}\left(x^2\text{e}^x\right) = \left[x^2 + 2nx + n(n-1)\right]\text{e}^x$ for $n \in \mathbb{Z}^+$. [7]

(b) Hence or otherwise, determine the Maclaurin series of $f(x) = x^2\text{e}^x$ in ascending powers of $x$, up to and including the term in $x^4$. [3]

(c) Hence or otherwise, determine the value of $\lim_{x \to 0}\left[\frac{\left(x^2\text{e}^x - x^2\right)^3}{x^9}\right]$. [4]

Do **not** write solutions on this page.

**12.** [Maximum mark: 22]

Consider the equation $(z-1)^3 = \mathrm{i}$, $z \in \mathbb{C}$. The roots of this equation are $\omega_1$, $\omega_2$ and $\omega_3$, where $\mathrm{Im}(\omega_2) > 0$ and $\mathrm{Im}(\omega_3) < 0$.

(a) (i) Verify that $\omega_1 = 1 + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$ is a root of this equation.

(ii) Find $\omega_2$ and $\omega_3$, expressing these in the form $a + \mathrm{e}^{\mathrm{i}\theta}$, where $a \in \mathbb{R}$ and $\theta > 0$. [6]

The roots $\omega_1$, $\omega_2$ and $\omega_3$ are represented by the points A, B and C respectively on an Argand diagram.

(b) Plot the points A, B and C on an Argand diagram. [4]

(c) Find AC. [3]

Consider the equation $(z-1)^3 = \mathrm{i}z^3$, $z \in \mathbb{C}$.

(d) By using de Moivre's theorem, show that $\alpha = \dfrac{1}{1 - \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}}$ is a root of this equation. [3]

(e) Determine the value of $\mathrm{Re}(\alpha)$. [6]

**References:**

Please **do not** write on this page.

Answers written on this page will not be marked.

Please **do not** write on this page.

Answers written on this page
will not be marked.